

Global Phase Diagram of a One-Dimensional Driven Lattice Gas

Dirk Helbing,^{1,2} David Mukamel,² and Gunter M. Schütz^{2,3}

¹ *II. Institute of Theoretical Physics, Pfaffenwaldring 57/III, 70550 Stuttgart, Germany*

² *Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot, Israel*

³ *Institut für Festkörperforschung, Forschungszentrum Jülich, 52425 Jülich, Germany*

We investigate the non-equilibrium stationary state of a translationally invariant one-dimensional driven lattice gas with short-range interactions. The phase diagram is found to exhibit a line of continuous transitions from a disordered phase to a phase with spontaneous symmetry breaking. At the phase transition the correlation length is infinite and density correlations decay algebraically. Depending on the parameters which define the dynamics, the transition either belongs to the universality class of directed percolation or to a universality class of a growth model which preserves the local minimal height. Consequences of some mappings to other models, including a parity-conserving branching-annihilation process are briefly discussed.

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The interplay of external driving fields and internal repulsive forces between particles can lead to interesting and unexpected phase transitions in the steady states of one-dimensional driven diffusive systems even if the interactions are only short-ranged [1]. Generically, the presence of boundaries or single defects in driven systems leads to shock waves and mutual blocking mechanisms which result in a breakdown of homogeneous particle flow. Thus localized static inhomogeneities are responsible for a variety of phenomena including first- and second-order phase transitions [2] or spontaneous symmetry breaking [3]. These observations are of practical importance for the qualitative understanding of many-body systems in which the dynamic degrees of freedom reduce to effectively one dimension as e.g. in traffic flow [4], kinetics of protein synthesis [5], gel-electrophoresis [6], or interface growth of thin films [7].

Whether continuous phase transitions can occur also in spatially *homogeneous* non-equilibrium systems in one dimension is less well-understood [8]. In particular, there is a long-standing conjecture [9] that in systems with local interactions the steady states have rapidly decaying correlations and, like in 1-d *equilibrium* models, no phase transition accompanied by algebraically decaying correlations takes place. On the other hand, recent studies of more complicated driven systems of three or more species of particles in 1d have demonstrated that phase separation may take place in these models [10], thus proving the possibility of long-range order, but leaving open the issue of continuous phase transitions with algebraic decay of correlations. In the absence of a general framework for studying non-equilibrium phase transitions, analyzing specific models could provide useful insight in these complex phenomena.

In this context, several translationally invariant one-dimensional growth models with local interactions which exhibit roughening transitions have recently been introduced. A common feature of these models is that one of the local transition rates which govern their dynamics is set to zero. The resulting roughening transition in

one class of models belongs to the universality class of directed percolation [11]. In another class of growth models which preserve the local minimal height, the transition is found to belong to a different universality class [12,13]. It would be of great interest to put these classes of models within a unifying framework, so that the various types of transitions, the associated crossover phenomena and the global phase diagram could be studied.

In this Letter we introduce a simple homogeneous driven 1d lattice gas model with local dynamics. It exhibits a phase transition where correlations decay algebraically and which is accompanied by spontaneous symmetry breaking. The model can be mapped onto a growth model where the transition becomes a roughening transition. By varying the parameters which define its dynamics, some types of the transitions discussed above can be realized. The various transitions and the global phase diagram are studied.

We consider a lattice gas which is an asymmetric exclusion process with next-nearest-neighbour interaction. Each lattice-site $i \in \{1, 2, \dots, L\}$ of a periodic chain may be either empty (\emptyset) or occupied by one particle of a single species, labeled A . The model evolves by random sequential updating. Particles hop to the right with constant attempt rate r (q) if the right nearest neighbour site is vacant and the nearest neighbour site at the left is occupied (empty). Unlike in the KLS-models [14], the left-hopping mechanism is different: A particle hops to the left with rate $p = 1 - q - r$ only if the next-nearest-neighbour site is empty as well. The model is therefore defined by the transitions

$$\begin{aligned} A A \emptyset &\rightarrow A \emptyset A \text{ with rate } r, \\ \emptyset A \emptyset &\rightarrow \emptyset \emptyset A \text{ with rate } q, \\ \emptyset \emptyset A &\rightarrow \emptyset A \emptyset \text{ with rate } p. \end{aligned} \tag{1}$$

By identifying vacancies with up-spins and particles with down-spins, these dynamics may be interpreted as a non-equilibrium spin-relaxation process. The choice $p = 0$ is a special case of the kinetic Ising models of Ref. [14], with

$r = q = 1/2$ corresponding to the totally asymmetric exclusion process (TASEP) [1]. In yet another mapping one obtains a growth model for a one-dimensional interface (see below).

Our interest is in the stationary behavior of the half-filled system, i.e. the asymptotic state of the system reached at very large times. A thorough survey of the phase diagram yields as main features a phase (*I*) with spontaneously broken Z_2 -symmetry between two antiferromagnetic stationary states and a disordered phase (*II*) (Fig. 1). As we shall argue below, the transition line separating the two phases belongs to the universality class of directed percolation except for $r = 0$, where the universality class is different.

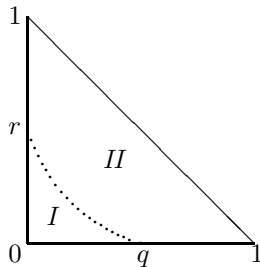


FIG. 1. Schematic phase diagram with second order phase transition line (dotted curve) between the disordered phase *II* and phase *I* with spontaneously broken Z_2 -symmetry and non-vanishing order parameter Δ (cf. Eq. (4)). At $r = 0$, there is a transition at $q = 1/2$ to phase separation with two ferromagnetically ordered areas and spontaneous breaking of translational invariance.

More specifically, we found that the stationary state can be calculated exactly along the four lines $r = 0$, $q = 0$, $q = 1/2$ and $p = 0$ (Fig. 1). (i) For $q = 1/2$, the system is disordered, and the stationary states are uncorrelated product measures. (ii) For $p = 0$, the stationary distribution is that of a one-dimensional Ising model [14]. Correlations are short-ranged with divergent correlation lengths only at the extremal points $q = 1$ (phase separation into regimes with complete ferromagnetic order but opposite magnetization) and $r = 1$ (complete antiferromagnetic order), respectively. (iii) The stationary state along $q = 0$ is also antiferromagnetically ordered, but at $r = p = 1/2$ there is an interesting phase transition in the dynamics of the system. Evidently, for small q , transitions between the two antiferromagnetic states $A\emptyset A\emptyset A\emptyset \dots$ and $\emptyset A\emptyset A\emptyset A \dots$ are possible with finite probability, if the system is *finite*. However, for $r > 1/2$ the flipping time between these two states diverges with a power law in system size L , whereas for $r < 1/2$ this flipping time diverges exponentially in system size. This is a signature for spontaneous symmetry breaking (and associated ergodicity breaking in the thermodynamic limit), even away from the line $q = 0$. (iv) Along $r = 0$ the minimal height of the corresponding growth model is conserved [12]. As in the related

class of models of Ref. [12], the dynamics satisfies detailed balance with respect to an energy functional which is proportional to the area under the interface. The point $p = q = 1/2$ (corresponding to a change in the sign of the energy E) marks the transition from an antiferromagnetic state to a state where complete phase ordering takes place and translational invariance is spontaneously broken. This transition is analogous to the wetting transition of Ref. [13].

This summary of exact results demonstrates the rich behavior that even rather simple homogeneous lattice gases may show and also indicates a certain degree of universality of these phenomena in 1D non-equilibrium systems. Here, we want to discuss the behavior of the system as it crosses the phase transition line between the broken symmetry phase *I* and the disordered phase *II*. We shall focus on the line $r = q$ with the limiting cases $r = q = 1/2$ (usual right hopping TASEP with uncorrelated disordered stationary state) and $r = q = 0$ (left hopping TASEP with next-nearest-neighbour repulsion and fully ordered stationary states). We performed Monte-Carlo simulations for half-filled periodic systems of size $L = 2^n$, mostly with $n = 10$. Expectation values were averaged over $4000L$ rounds after a transient period of at least the same duration.

We study the quantity

$$\Delta(t) = \frac{1}{t} \int_0^t dt \frac{2}{L} \sum_{i=1}^L (-1)^i \langle n_i(t) \rangle. \quad (2)$$

where $n_i = 0$ corresponds to an empty site i and $n_i = 1$ to an occupied one. In the limit $t \rightarrow \infty$, it corresponds to the non-conserved order parameter $2/L \sum_i (-1)^i \langle n_i \rangle$, which is the stationary difference in sublattice particle densities (the ‘staggered magnetization’ in spin language). Because of ergodicity, the stationary value of the order parameter in a *finite* system vanishes by symmetry. However, as a signature of spontaneously broken symmetry in the thermodynamic limit, one expects an initial decay to some quasi-stationary value Δ_0 , before Δ eventually approaches zero for very long times (exponentially large in system size). On the other hand, in the disordered phase one expects an initially ordered state with $\Delta = 1$ to rapidly disorder, i.e. one expects Δ to decay quickly to zero.

A second quantity of interest is the stationary particle current which, according to the definition (1) of the process, on the line $r = q = (1 - p)/2$ is given by $j(q) = q \langle n_i(1 - n_{i+1}) \rangle - (1 - 2q) \langle (1 - n_{i-1})(1 - n_i)n_{i+1} \rangle$. Clearly, $j(0) = 0$ and $j(1/2) = 1/8$, up to a small finite-size correction of order $1/L$. The presence of spontaneous symmetry breaking suggests $j = 0$ for all $q \leq q_c$ (up to exponentially small corrections in system size), since any finite current would lead to a transition between the two degenerate stationary states with $\Delta = \pm \Delta_0$ within a finite time.

This intuitive picture is well-supported by our Monte Carlo simulations (Fig. 2). The current j vanishes in phase *I* and the order parameter Δ_0 vanishes in phase *II*. We find a phase transition point $q_c = 0.1515 \pm 0.0005$ for $r = q$, above which the current decays with a power law

$$j \sim (q - q_c)^y \quad (3)$$

with $y \approx 1.7 \pm 0.1$. Approaching the critical point q_c from below, Δ_0 decays with a power law

$$\Delta_0 \sim (q_c - q)^\theta \quad (4)$$

with $\theta \approx 0.54 \pm 0.04$. To investigate whether this continuous bulk phase transition is accompanied by spatial long-range order—as one would expect in an equilibrium system—we examine the stationary density correlation function $C(k) = 4(\langle n_i n_{i+k} \rangle - \langle n_i \rangle \langle n_{i+k} \rangle)$ which turns out to decay to a non-zero value below q_c . At the critical point, correlations decay algebraically

$$C(k) \sim k^{-\gamma}, \quad (5)$$

where $\gamma \approx 1.0 \pm 0.1$ (Fig. 2) [16].

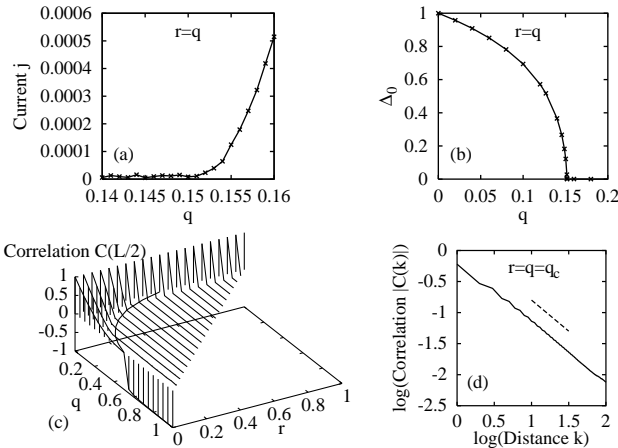


FIG. 2. (a) Stationary current and (b) order parameter along the line $r = q$. Below: Correlation function $C(k)$ (c) for maximal distance $k = L/2$ as a function of r and q and (d) as a function of k for $r = q = q_c$. The dashed line corresponds to a slope of -1 . The measured data are connected by straight lines as a guide for the eye.

We can gain further insight by considering the mapping to an interface model [15] which is described by height difference variables $1 - 2n_i$ and an additional stochastic variable h , representing the absolute height of the interface at some reference point. Each time a particle hops to the right, the local height increases by two units (deposition), whereas hopping to the left describes a height decrease (evaporation) (Fig. 3). Thus the current gives the stationary growth velocity of the interface, while the density correlation function measures height-gradient correlations. Growth occurs at local minima

with rate q , independently of the precise nature of the immediate environment. However, evaporation of particles does not occur from a “flat” part of the interface: The corresponding process $A\emptyset A\emptyset \rightarrow A A\emptyset\emptyset$ is forbidden. On a coarse-grained scale this means that in a locally flat piece of the interface evaporation is not strong enough to create little craters which could then further grow. A similar situation (with different microscopic dynamics) was investigated by Alon et al. [11], who found a phase transition between a smooth phase where no current flows and a rough, growing phase in the universality class of the KPZ-equation. The transition is related to directed percolation in 1+1 dimensions and is accompanied by spontaneous symmetry breaking in the height variable h .

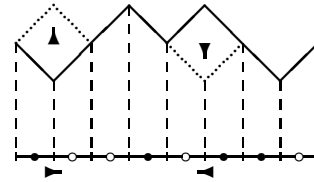


FIG. 3. Mapping between lattice gas dynamics and interface growth in 1+1 dimensions. A positive (negative) unit slope belongs to a vacancy (particle). The interface flips (vertical arrows) correspond to particles hopping on the lattice (horizontal arrows).

Here we find similar behavior which is most transparent in the two limiting cases $q = 0$ and $q = 1/2$, respectively. The limit $q \rightarrow 1/2$ corresponds to the TASEP (growing, rough interface), which indeed describes interface growth in the KPZ universality class [17]. In the limit $q \rightarrow 0$, there is no current and one has spontaneous symmetry breaking between (macroscopically) flat interfaces on an even or odd height level, respectively. We stress, however, that spontaneous symmetry breaking occurs already on the level of the particle description, i.e. without reference to the extra height variable. Assuming universality, one expects [11] the exponent y to be given by the critical exponent $\nu_{\parallel} \approx 1.73$ of the DP-coherence time [18] and also a logarithmic divergence of the interface width $w = [L^{-1} \sum_i (h_i - L^{-1} \sum_i h_i)^2]^{1/2}$. This is in agreement with our results in Eq. (3) and Fig. 4. Also the value (4) of the order parameter exponent θ is consistent with the result $\theta = 0.55 \pm 0.05$ reported in Ref. [11], thus independently confirming universality. Results on the correlation exponent γ have not been reported in earlier work.

The transition at $r = 0$ is of a different nature. Here the model satisfies detailed balance and the current vanishes both above and below the transition. At the phase transition point $q_c = 1/2$ the lattice gas is uncorrelated. Using the interface representation of the model one can show [13] that the interface width diverges algebraically with an exponent $1/3$ as q approaches $1/2$ from below (Fig. 4).

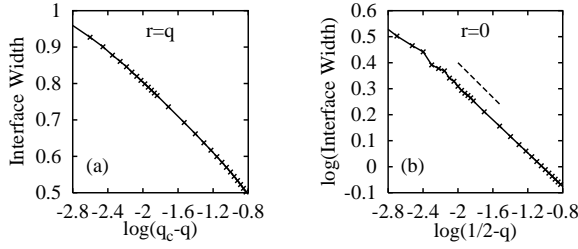


FIG. 4. Interface width close to the phase transition points along (a) $r = q$, suggesting a logarithmic-like divergence, and (b) $r = 0$, respectively. The slope of the dashed line is $-1/3$.

The understanding of θ and of the new correlation exponent γ (which have no conventional interpretation within the framework of directed percolation), and the behavior of these two quantities at the transition at $r = 0$ have to be addressed in future work. Also the behavior of the system away from half-filling, where preliminary results suggest the disappearance of phase I , is an open issue. Returning to our original question we conclude at this point that the stationary states of homogeneous one-dimensional lattice gas models may exhibit continuous bulk phase transitions with an algebraic decay of correlations even if interactions are short-ranged. In our model, this transition results from dynamical constraints which—unlike in the KLS models—lead to a competition between a disordering dynamics (the right-hopping process) and processes forcing the system into either of two antiferromagnetically ordered states (the restricted left hopping process). For sufficiently strong ordering processes, the stationary current ceases to flow and spontaneous symmetry breaking sets in.

It is interesting to consider yet another mapping of our model, obtained by mapping particles into vacancies and vice versa on one (either even or odd) sublattice. The resulting dynamics are that of a new class of parity-conserving (PC) branching-annihilation processes $\emptyset AA \rightleftharpoons \emptyset\emptyset\emptyset$ and $A\emptyset\emptyset \rightleftharpoons AAA$ with no absorbing state. In addition to particle-parity conservation (particle number modulo 2), there is a $U(1)$ symmetry which results from the particle number conservation of the original hopping process. Generically, one expects parity-conserving branching-annihilation processes not to be in the DP universality class, but in a distinct PC universality class [19]. From our results it appears that, in the presence of additional symmetries, the picture of phase transitions in 1D branching-annihilation processes is more complicated.

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- [1] B. Schmittmann and R.K.P. Zia, in *Phase Transitions and Critical Phenomena*, eds. C Domb and J. Lebowitz (Academic, London, 1995), Vol. 17.
- [2] J. Krug, Phys. Rev. Lett. **67**, 1882 (1991); G. Schütz and E. Domany, J. Stat. Phys. **72**, 277 (1993); B. Derrida, M.R. Evans, V. Hakim, and V. Pasquier, J. Phys. A **26**, 1493 (1993); A.B. Kolomeisky, G.M. Schütz, E.B. Kolomeisky, and J.P. Straley, J. Phys. A **31**, 6911 (1998).
- [3] M.R. Evans, D.P. Foster, C. Godrèche, and D. Mukamel, Phys. Rev. Lett. **74**, 208 (1995); J. Stat. Phys. **80**, 69 (1995); C. Godrèche, J.M. Luck, M.R. Evans, D. Mukamel, S. Sandow, and E.R. Speer, J. Phys. A **28**, 6039 (1995).
- [4] M. Schreckenberg, A. Schadschneider, K. Nagel and N. Ito, Phys. Rev. E **51**, 2939 (1995); D. Helbing and M. Schreckenberg (unpublished); S. Yukawa, M. Kikuchi, and S. Tadaki, J. Phys. Soc. Jap. **63**, 3609 (1994); T. Nagatani, J. Phys. A **28**, 7079 (1995).
- [5] J.T. MacDonald, J.H. Gibbs, and A.C. Pipkin, Biopolymers **6**, 1 (1968); G.M. Schütz, Int. J. Mod. Phys. B **11**, 197 (1997).
- [6] D.P. Aalberts and J.M.J. van Leeuwen, Electrophoresis **17**, 1003 (1996).
- [7] T.D. Blake and K.J. Ruschak, Nature **282**, 489 (1979).
- [8] Phase transitions in homogeneous one-dimensional nonequilibrium systems have usually been observed in models with absorbing states, as, e.g., the “dry” state in directed percolation models [E. Domany and W. Kinzel, Phys. Rev. Lett. **53**, 311 (1984); W. Kinzel, Z. Phys. B **58**, 229 (1985)]. Here, our interest is in driven lattice gases without absorbing states.
- [9] H. Spohn, *Large Scale Dynamics of Interacting Particles* (Springer, Berlin, 1991).
- [10] M.R. Evans, Y. Kafri, H.M. Kudovaly and D. Mukamel, Phys. Rev. Lett. **80**, 425 (1998); P.F. Arndt, T. Heinzl and V. Rittenberg, J. Phys. A **31**, 833 (1998).
- [11] U. Alon, M.R. Evans, H. Hinrichsen, and D. Mukamel, Phys. Rev. Lett. **76**, 2746 (1996) and Phys. Rev. E **57**, 4997 (1998).
- [12] H.M. Kudovaly and D. Dhar, J. Stat. Phys. **90**, 57 (1998).
- [13] H. Hinrichsen, R. Livi, D. Mukamel, and A. Politi, Phys. Rev. Lett. **79**, 2710 (1997).
- [14] S. Katz, J.L. Lebowitz and H. Spohn, J. Stat. Phys. **34**, 497 (1984).
- [15] P. Meakin, P. Ramanlal, L. Sander, and R. Ball, Phys. Rev. A **34**, 5091 (1986); M. Plischke, Z. Rácz, and D. Liu, Phys. Rev. B **35**, 3485 (1987).
- [16] The errors for the exponents are estimated from the slopes of j , Δ_0 , and $C(k)$, respectively, in double-logarithmic scale by using the two values $q_c = 0.152$ and $q_c = 0.151$, respectively, for the numerically determined critical point, and using data for the largest available system size $L = 1024$. Finite-size effects make a more precise determination of the critical point difficult.
- [17] J. Krug and H. Spohn, in: *Solids far from Equilibrium*, edited by C. Godreche, (Cambridge University Press, Cambridge, 1991), and references therein.
- [18] W. Kinzel, in *Percolation Structures and Processes*, edited by G. Deutscher, R. Zallen and J. Adler, Annals of the Israeli Society, Vol. 5 (Hilger, Bristol, 1983).
- [19] J. Cardy and U. Täuber, J. Stat. Phys. **90**, 1 (1998).

